PHYSICAL REVIEW E **67**, 035102(R) (2003)

Interface motion and pinning in small-world networks

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(Received 15 October 2002; published 20 March 2003)

We show that the nonequilibrium dynamics of systems with many interacting elements located on a small-world network can be much slower than on regular networks. As an example, we study the phase ordering dynamics of the Ising model on a Watts-Strogatz network, after a quench in the ferromagnetic phase at zero temperature. In one and two dimensions, small-world features produce dynamically frozen configurations, disordered at large length scales, analogous to random field models. This picture differs from the common knowledge (supported by equilibrium results) that ferromagnetic shortcut connections favor order and uniformity. We briefly discuss some implications of these results regarding the dynamics of social changes.

DOI: 10.1103/PhysRevE.67.035102 PACS number(s): 89.75.Hc, 05.50.+q, 05.70.Ln, 64.60.Cn

Small-world networks have received a great deal of attention in the past few years, in particular for their realistic description of the topology of the interactions that take place among populations in various biological, social, or economical systems [1]. An important feature of small worlds, which is not shared either by regular lattices or random networks, is the interplay that exists between local (or '' physical'') interactions, e.g., between nearest neighbors, and nonlocal ones, involving nodes (or agents) separated by large distances but connected through shortcuts. Among other outstanding topological properties, the effective space dimension of such networks grows linearly with their size [2], even if the fraction of sites with shortcuts is very small.

The strong connectivity of small worlds usually enhances dramatically cooperative effects, as predicted by epidemic models of spreading of diseases [4], or of propagation of conventions or rumors in social systems [5]. Naturally, many models of social dynamics have been inspired from the Ising model [6]. The Ising model on a small world exhibits ferromagnetic order at low temperatures even in one dimension (1D) [7], while, in higher dimensions, the critical temperature is increased compared with that of the regular lattice [8]. In addition, the fact that the transition is of mean-field nature agrees with the intuitive argument that each site is effectively close to a large number of sites due to the shortcuts of the lattice.

However, because of their inherent random topology, one may ask whether in some situations small-world networks would not rather exhibit features characteristic of disordered systems. In this work, we study as a basic example the nonequilibrium dynamics of the Ising model, as observed after a rapid quench from the high-temperature phase to the ferromagnetic phase. We show that the random (all ferromagnetic) connections that enhance ordered states at thermodynamic equilibrium are responsible in the present case for very slow dynamics and stabilize at large times configurations that, instead of being uniform, are spatially heterogeneous. At zero temperature, systems do not perform long range order dynamically, but remain asymptotically trapped in metastable states characterized by a finite domain size. These features are reminiscent of nonequilibrium processes in the random field Ising model (RFIM) [9], in binary mixtures with fixed impurities [10], as well as in a few social models on regular lattices like the voter model [11]. This has to be contrasted with the much more efficient phase ordering kinetics of the Ising model on regular lattices (or Model *A* in the lexicon of Hohenberg and Halperin [12]), where the mean size of ordered domains grows with time as $t^{1/2}$ [13]. Our present analysis focuses on the motion of domain walls between '' up'' and '' down'' domains, and shows evidence of competing effects between surface tension and pinning (or localizing) effects.

We use a standard model of small-world network [14] consisting of a regular square lattice (or a chain in 1D) composed of N nodes connected to their nearest neighbors. For each site, we then establish with a probability p an additional connection, or shortcut, linking the considered site to another site chosen at random in the lattice. (We do not remove the nearest neighbors connections.) For $p=0$, the lattice is regular, while for $p=1$, the network is strongly disordered. Here, we will consider only the so-called ''small-world'' limit, which corresponds to the case $p \ll 1$, where connections are mainly local and only long ranged for a small fraction of nodes.

On a fixed network, we then assign to each node a spinlike variable $S_i = \pm 1$: it represents a social convention, initially chosen at random for each node. At each time step, each node updates its convention in order to reach a better consensus with the nodes it is connected to. In other words, the system follows a zero-temperature Glauber dynamics with the Hamiltonian $H = -\Sigma_{\langle i,j \rangle} J_{ij} S_i S_j$, where the sum is performed over all possible pairs of nodes. $J_{ij}=1$ if sites i and j are connected, $J_{ij}=0$ otherwise. At each step, a spin is thus chosen at random and flipped. If H decreases, does not change, or increases, the new configuration is accepted with probability 1, 1/2, and 0, respectively.

In regular networks ($p=0$), the system evolves toward a minimum of H (all S_i's equal to $+1$ or -1). Transient configurations are characterized by the presence of growing and competing ordered domains of '' up'' and '' down'' spins. The large time dynamics is controlled by the motion and annihilation of interfaces (or domain walls) that separate these domains. As for many other systems ordering in phases with broken symmetries, its time evolution is self-similar: The two-point correlation function, $C(r,t) = \langle S_i(t) S_{i+r}(t) \rangle$, obeys a scaling relation $C(r,t) = f(r/\xi(t))$, where f is a scal-

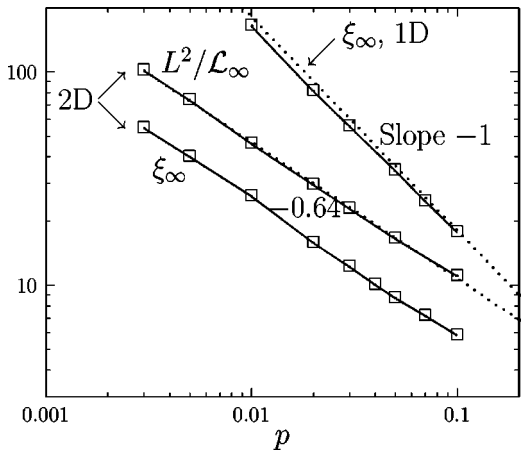

FIG. 1. Asymptotic correlation length ξ_0 in lattice spacing units as a function of the reconnection probability p, in 1D and 2D. $L^2/\mathcal{L}_\infty$ is the interface characteristic length in 2D.

ing function, while ξ, the domain size, grows as $t^{1/2}$ [13].

On the contrary, in small-world networks ($p \neq 0$, $p \ll 1$) one observes after some time that the typical domain size of ordered spins saturates to a finite value, which decreases when the density of shortcuts (p) increases. For a one dimensional chain (of length $L=10^5$), we plot $\xi(t=\infty)$ as a function of p in Fig. 1. The correlation length is determined from the half width of C averaged over ten networks and initial conditions. A behavior $\xi(t=\infty) \propto 1/p$ can be observed. $1/p$ represents the characteristic size of the one dimensional network, i.e., the average distance between two nodes that have long range connections (or ''influent'' nodes).

Influent sites strongly affect the motion of interfaces. At low p, most of these nodes are characterized by one additional connection. In Fig. 2(a), two nodes far apart, A and B, are connected, and $S_A = -S_B$. Any interface I passing through node A leftward cannot jump back toward the right, since it is energetically unfavorable. Therefore, at large times, through interface motion, influent nodes will tend to be (irreversibly) connected to nodes that have the same spin (a situation analogous to assortative mixing, as observed in some real life networks [3]).

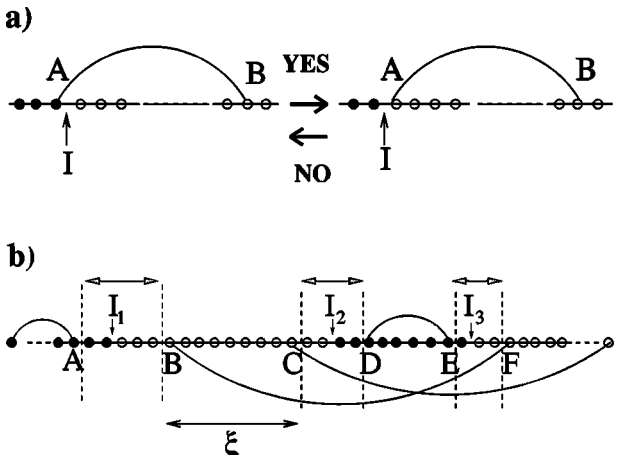

FIG. 2. Up and down domains. Domain walls I_n become localized in 1D.

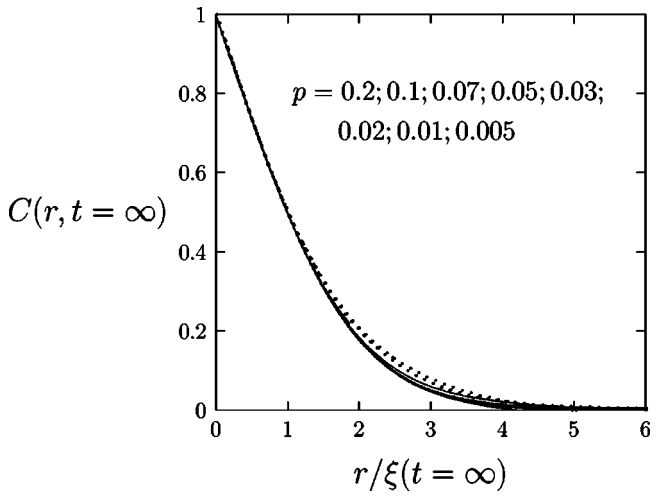

FIG. 3. Two-point correlation function in 1D (dotted lines) and 2D (solid lines) as a function of r/ξ_∞, for various p.

This argument can be extended to a succession of domains. Figure 2(b) illustrates a typical large time configuration. In this example, interface I_1 stands between two influent nodes A and B with opposite spins: for the reason mentioned above, I_1 cannot jump to the left of A, nor to the right of B. The interface is then localized, i.e., restricted to perform a random walk within the interval $[A, B]$. Interface I_1 is therefore unable to annihilate with interface I_2, which is localized between C and D. Hence, the two disjoint black domains cannot merge to form a bigger one, what would happen in the standard Ising model. The domain size, or correlation length ξ, does not exceed the distance, of order $1/p$, that separates influent unlike (antagonist) successive nodes.

We also find that the structure of frozen configurations obeys a scaling relation with p, i.e., that they are statistically independent of p via proper length rescaling. The asymptotic correlation function $C(r, t=\infty)$ is plotted in Fig. 3 as a function of the reduced variable $r/\xi_\infty(p)$, for various values of p. Data collapse rather well on a single curve. At short times, the kinetics is not affected by the small-world structure of the lattice, and that $\xi(t)$ starts growing as $t^{1/2}$. When interfaces become localized, the structure can be roughly seen as the one given by the standard Glauber dynamics of the one dimensional Ising model stopped at a time $(1/p)^2$. Since that problem obeys dynamical scaling, frozen configuration should scale with parameter p. However, this picture is not quantitatively correct, as the scaling functions in both problems slightly differ.

Finite, low-temperature effects are quite subtle in one dimensional small worlds since they do not destroy the ferromagnetic order observed at $T=0$, unlike in usual Ising chains. One can interpret here the order-disorder transition temperature T_c as the temperature where ordering via interface jumps over localizing barriers (which enable further domain merging) and no longer overcomes disordering happening within domains (subdomain creation). In Fig. 2(b), interface I_2 (or I_1) can jump in interval $[B, C]$ at a rate $r_a = \exp(-2/T)$. Besides, the rate at which any spin among the

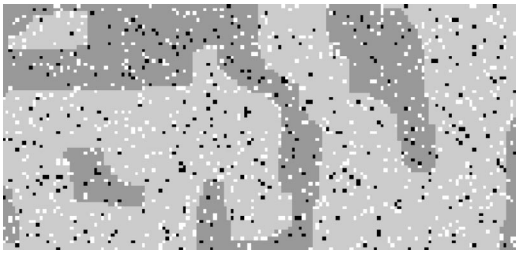

FIG. 4. Frozen domains (in gray) for $p=0.05$ (6% of the system's total area). The white and black dots represent the ''like'' and ''unlike'' influent nodes, respectively.

p^{-1} spins of interval $[D, E]$ would flip is $r_b=p^{-1}\exp(-4/T)$: it is roughly the rate at which a white domain is created and can start to grow. Qualitatively, the order-disorder transition occurs when $r_a=r_b$. This gives: $T_c\simeq -2/\ln p$, an expression derived (with a $\propto$ sign) in Ref. [7] using the replica method. The above relation may be exact as $p\to 0$ (as the numerical prefactors in the different rates become irrelevant). Simulation results (not shown) give $T_c=-2.3/\ln p$ for $p=0.03$.

In two dimensions, one also observes that random initial configurations freeze at large times. Figure 1 displays as a function of p the asymptotic correlation length (determined from $C(r,t=\infty)$ averaged over eight networks with 1500^2 spins), as well as the length associated with interface density. The latter is defined as $L^2/\mathcal{L}$, where L is the system's linear extent and $\mathcal{L}$ the total length of all boundaries. Both length scales remain proportional to each other when varying p, suggesting that frozen configurations can be characterized by one characteristic length scale, referred to as the ''domain size,'' $R_\infty(p)$. Numerical results suggest that R_∞ varies as an inverse power law of p, with a nontrivial exponent close to $-2/3$ over nearly two decades. Surprisingly, R_∞ does not scale as $p^{-1/2}$, the characteristic length scale of the network [2]. Once again, the spin-spin correlation function at $t=\infty$ scales rather well as $C(r)=f(r/\xi_\infty(p))$, see Fig. 3.

Figure 4 shows a typical frozen pattern at $p=0.05$. The positions of the ''influent'' spins are marked by dots: in white, those that are connected to another influent spin of the same sign (''like'' pairs, of number density n_l), in black those connected to a spin of opposite sign (''unlike'' pairs, number density n_u). Initially, $n_l\simeq n_u$, but as coarsening proceeds, ''unlike'' dots turn more easily to '' like'' than the contrary, as in one dimension [Fig. 2(a)]. Once again, mixing tends to be assortative ($n_l>n_u$), but with the increase of n_l, at some point, there are no more possible moves toward better consensus. We find numerically that coarsening stops and interfaces get pinned when $n_l\simeq 1.86 n_u$.

The finite domain size can be interpreted as the result of competing effects between surface tension (the driving force for domain growth) and energy barriers created by the multiplication of influent ''like'' sites. We picture the system as a collection of L^2/R^2 domains of radius R, and estimate its energy change when domains coarsen from R to $R+dR(dR>0)$. The usual contribution from surface tension is $\delta E_I\propto -2L^2dR/R^2$. Meanwhile, the number of influent nodes that flip spin is proportional to $2pRdR(L^2/R^2)$.

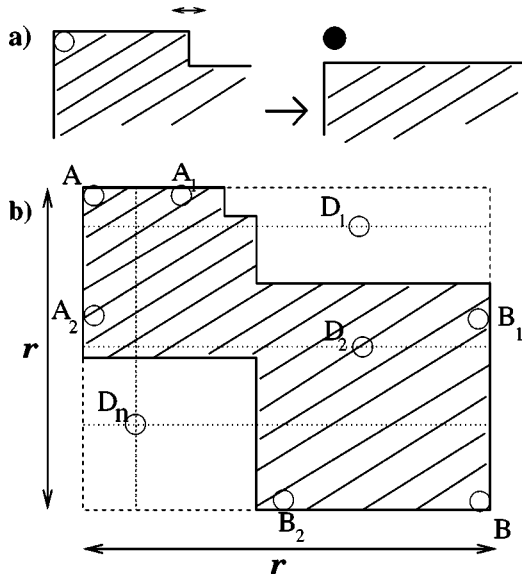

FIG. 5. (a) Free and (b) pinned domains in the presence of ''like'' influent nodes.

'' Like'' nodes turn to '' unlike'' (with an energy cost per spin of 2), and reversely (with an energy decrease of -2). The total energy difference thus reads

$$\delta E\propto\left[-\frac{2}{R^2}+\frac{n_l-n_u}{n_l+n_u}\frac{4p}{R}\right]L^2dR. \tag{1}$$

The second term is positive and dominates at large R. Hence, coarsening is arrested when $\delta E=0$, or $R_\infty\sim p^{-1}$. This argument is somehow similar to the (equilibrium) Imry-Ma argument for the RFIM [15]. Yet, an important difference is that here the average magnetic field felt on influent nodes (or ''impurities'') is not zero, but has been biased ($n_l\neq n_u$) due to previous spin flips.

The above continuous Imry-Ma-like argument qualitatively explains frozen states, but overestimates R_∞ ($\sim p^{-1}$ instead of $p^{-2/3}$). The exponent $-2/3$ can be explained as an effect of the square lattice. As shown in Fig. 5(a), a single influent ''like'' node located at a domain corner can disappear through the diffusive motion of a step. Figure 5(b) represents then the simplest distribution of ''like'' nodes such that the hatched domain cannot shrink. It is composed of two right-angle corners $\{A,A_1,A_2\}$ and $\{B,B_1,B_2\}$ defining a square $r\times r$. If the other white nodes $\{D_1,\ldots D_n\}$ contained in the square do not form any right-angle corners, then this region encloses the smallest (or '' critical'') pinned domain: any bubble of hatched region contained in the square which does not contain both corners $\{A,A_1,A_2\}$ and $\{B,B_1,B_2\}$ will shrink. Any larger bubble will not. We now calculate the probability $P_{\text{freeze}}(r)$ that a configuration such as represented in Fig. 5(b) has a size r, and then identify r^* such that $P_{\text{freeze}}(r^*)$ is maximal with the asymptotic domain size R_∞ in the disordered medium.

Given the node A located at the origin, the probability that there is at least one white dot (A_1) on the same line within a distance r is $P_1(r)=1-(1-p_l)^r$, with $p_l/p=n_l/(n_l+n_u)$

the fraction of influent nodes that are ''like'' (here, the numerical value of this ratio—close to 0.65—is unimportant and could be set to 1). Therefore, $P_{\text{freeze}}(r) = p_l[P_1(r)]^4 P_2(r)$, with P_2 the probability that the D_n's do not form right-angle corners, i.e., that each node D_i is at least located on a line or a column not occupied by an other D_j [see the dotted lines in Fig. 5(b)]. P_2 can be approximated as

$$P_2(r) \simeq \sum_{n=0}^{(r-2)^2} (1-p_l)^{(r-2)^2-n} p_l^n C_{(r-2)^2}^n [1-P_1(r)]^n, \tag{2}$$

or $P_2(r) \simeq [1-p_l P_1(r)]^{(r-2)^2}$. In the sum (2) we have multiplied the probability of having n white dots inside the square by the probability $[1-P_1(r)]^n$ that n independent dots have no neighbors on the same line within r. For n small, no or few corners can be formed anyway, so that relation (2) slightly underestimated P_2, since a small fraction of empty sites are counted twice ($[1-P_1(r)]^n \lesssim 1$). For n large, on the contrary, relation (2) overestimates P_2, since it is impossible to locate many dots without forming corners (while $[1-P_1(r)]^n$ is small but $\neq 0$). We suppose that both errors compensate. This factorization enable us to compute the most probable square size r^* analytically. $P_1(r)$ increases with r, $P_2(r)$ decreases with r, and $P_{\text{freeze}}(r)$ has one single maximum. Assuming $p \ll 1$, $r \gg 1$, $rp \ll 1$, we find that $r^* = p_l^{-2/3} \propto p^{-2/3}$, in agreement with the numerical results.

To summarize, we have shown with an example that assortative mixing in small-world networks can dynamically generate frozen metastable states. At large times, some influent nodes have simply no immediate interest to evolve. These results suggest that long term dynamics in highly connected social systems can produce spatial heterogeneities (or segregation), despite that these configurations are not the most desired ones by individual agents. A similar picture, in agreement with some empirical observations, was drawn recently from antiferromagnetic models on scale-free networks [16]. Right after strong political changes (in Eastern European countries in 1989, in Mexico in 2000) the evolution of reforms can be fast, but social inertia takes over rapidly and renders further adjustments difficult or null. Physically speaking, the response of social systems to external forcings (i.e., large-scale policies) is susceptible to exhibit some of the interesting features known for disordered systems [9]. While revising the manuscript, we became aware of a similar study on the voter model on small worlds [17].

We acknowledge fruitful discussions with G. Cocho, J. Viñals, and R. Boyer.

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